Special Class of Nonlinear Damping Models in Flexible Space Structures

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A special class of nonlinear damping models is investigated in which the damping force is proportional to the product of positive integer or the fractional power of the absolute values of displacement and velocity. For a one-degree-of-freedom system, the classical Krylov-Bogoliubov "averaging" method is used, whereas for a distributed system, both an ad hoc perturbation technique and the finite difference method are employed to study the effects of nonlinear damping. The results are compared with linear viscous damping models. The amplitude decrement of free vibration for a single mode system with nonlinear models depends not only on the damping ratio but also on the initial amplitude, the time to measure the response, the frequency of the system, and the powers of displacement and velocity. For the distributed system, the action of nonlinear damping is found to reduce the energy of the system and to pass energy to lower modes.

I. Introduction

ONE of the major challenges remaining in the development of large space structures is to determine a damping mechanism in order to stabilize flexible flight structures such as solar arrays, antennas, and platforms. As the size and flexibility of space structures increase, the need to characterize energy dissipation in a more appropriate and accurate manner also increases. Under the assumption of linear viscous damping, the amplitude decrement of free vibration depends only on the damping ratio, regardless of what the frequency or initial conditions might be. Numerous experimental results, such as those in the Spacecraft Control Laboratory Experiment (SCOLE),[1,2] indicate that this is far from sufficient and that there is a great need for understanding the damping mechanism that may be inherently nonlinear.

Various nonlinear models, such as linear dampers with clearance, Coulomb friction dampers, velocity-nth power damping, and so forth, have been investigated in the past.[3–5] In many cases, these models can be represented by a damping force that is proportional to the product of integer or fractional powers of the absolute values of displacement and velocity. Balakrishnan introduced this nonlinear model in Ref. 6 and obtained approximate solutions using the Krylov-Bogoliubov "averaging" method.[7] He also showed that these results can be quite useful to study the response of flexible structures to nonlinear boundary feedback control. In this paper we further study this special class of nonlinear damping models. We use the Krylov-Bogoliubov averaging technique for a one-degree-of-freedom system and employ both an ad hoc perturbation method and a finite difference technique for a distributed system.

This paper is organized as follows. In Sec. II, the approximate equations of amplitude are derived for a single-degree-of-freedom system with nonlinear damping. In Sec. III, the transient response of free vibration of a single-degree-of-freedom nonlinear system is compared to that of a system with linear viscous damping. In Sec. IV, the perturbation solutions are derived for the vibration of a pinned-pinned beam with nonlinear damping. In Sec. V, the vibration of the pinned-pinned beam with nonlinear damping is simulated via finite difference methods, and the results are compared with those obtained using the perturbation solution discussed in Sec. IV.

II. Single-Degree-of-Freedom System

The classical Krylov-Bogoliubov averaging method, introduced in 1947, is basically a method of variation of parameters. Over the decades, this averaging technique has been employed to study nonlinear mechanics, and solutions can be found in the literature[8] for special cases of nonlinear differential equations. Balakrishnan[6] applied this averaging method to a particular class of nonlinear damping models that will be discussed in detail in this section. Since this damping model is representative of a variety of nonlinear damping mechanisms, we further study the effects of the model on vibrating structures of the special class of nonlinear damping represented as

$$m\ddot{x} + c|x|^a|\dot{x}|^b\dot{x} + kx = 0 \tag{1a}$$

or

$$\ddot{x} + \gamma|x|^a|\dot{x}|^b\dot{x} + \omega^2 x = 0 \tag{1b}$$

$$x(0) = A_0, \qquad \text{and} \qquad \dot{x}(0) = 0 \tag{1c}$$

where $\gamma = c/m$ and $\omega^2 = k/m$, c is damping constant, and $a, b > 0$.

Note that the term $\gamma|x|^a|\dot{x}|^b\dot{x}$ represents the dissipating effect of a nonlinear damper with a and b both being positive integers or fractions.

When $\gamma \ll 1$ (c small relative to m), we may apply the averaging method of Krylov-Bogoliubov[7] to obtain an ap-

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proximate equation

$$x(t) = A(t)\sin[\omega t + \phi(t)]$$

where the amplitude $A(t)$ and phase angle $\phi(t)$ satisfy the following equations

$$\frac{dA(t)}{dt} = -\frac{\gamma}{\omega}K_0(t) \tag{2a}$$

and

$$\frac{d\phi(t)}{dt} = \frac{\gamma}{\omega A}P_0(t) \tag{2b}$$

The functions $K_0(t)$ and $P_0(t)$ in Eqs. (2a) and (2b) are defined as

$$K_0(t) = \frac{1}{2\pi}\int_0^{2\pi} D(A\sin\phi, A\omega\cos\phi)\cos\phi\, d\phi \tag{3a}$$

$$P_0(t) = \frac{1}{2\pi}\int_0^{2\pi} D(A\sin\phi, A\omega\cos\phi)\sin\phi\, d\phi \tag{3b}$$

where $D(x,\dot{x})$ is equal to $|x|^a|\dot{x}|^b$ for the choice of nonlinear damper in Eqs. (1a) and (1b), that is $D(x,\dot{x}) \triangleq D(A\sin\phi, A\omega\cos\phi)$. Substituting $D(x,\dot{x})$ into Eqs. (3a) and (3b), we obtain

$$K_0(A) = \omega^{b+1}A^{a+b+1}\mu \tag{4a}$$

$$P_0(A) = 0 \tag{4b}$$

where

$$\mu = \frac{1}{2\pi}\int_0^{2\pi} |\sin\phi|^a|\cos\phi|^{b+2}\, d\phi \tag{4c}$$

Equation (4b) implies that for the choice of a nonlinear damper represented by $\gamma|x|^a|\dot{x}|^b\dot{x}$, the phase angle $\phi(t)$ does not, on the average, change over time.

The positive number μ in Eq. (4c) is called the nonlinear damping factor. Be changing variables in Eqs. (4c), it can be shown that

$$\mu = \frac{2}{\pi}\int_0^{\pi/2} |\sin\phi|^a|\cos\phi|^{b+2}\, d\phi \tag{4d}$$

The preceding expression for μ is very similar to that for the so-called "damping force-amplitude ratio γ_n" proposed by Jacobsen[9] when studying equivalent viscous damping. By employing the properties of the gamma function, it is found that

$$\mu = \frac{\Gamma[(a+1)/2]\Gamma[(b+3)/2]}{\pi\Gamma\{[(a+b)/2]+2\}} \tag{4e}$$

where $\Gamma(\cdot)$ represents the gamma function. Furthermore, when a is an odd number, it can be shown that

$$\mu = \frac{2(n+1)n!}{(b+3)(b+5)\cdots(b+3+2n)} \qquad \text{for any } b > 0$$

With the preceding information, we are in a position to derive equations for the amplitude $A(t)$ and displacement $x(t)$ for a system with nonlinear damping.

For the nonlinear damping system, $(a+b>0)$, we substitute Eq. (4a) into Eq. (2a) to obtain a differential equation for the amplitude $A(t)$

$$\frac{dA(t)}{dt} = -\gamma\mu\omega^b A(t)^{a+b+1}$$

It can be shown that

$$A(t) = \left[\frac{m}{c\mu(a+b)\omega^b(t+t_c)}\right]^{1/(a+b)} \tag{5a}$$

where

$$t_c = \frac{m}{c\mu\omega^b(a+b)A_0^{a+b}} \tag{5b}$$

$$x(t) = \left[\frac{m}{c\mu(a+b)\omega^b(t+t_c)}\right]^{1/(a+b)}\cos\omega t \tag{5c}$$

Recall that for linear damping viscous damping $(a+b=0)$ (Ref. 10)

$$x(t) = A_0 e^{-\zeta\omega t}\cos\omega t$$

In Eq. (5c), the quantity t_c is associated with the initial conditions but is not the initial time. The constant t_c has units of time and is never equal to zero.

Defining $t + t_c \triangleq n(2\pi/\omega)$ and replacing $\hat{A}(n(2\pi/\omega) - t_c)$ by $\hat{A}(n)$ in Eq. (5a), we obtain an expression for the amplitude in terms of the number of cycles of nonlinear vibration

$$\hat{A}(n) = \left[\frac{m^{(b+1)/2}}{c\mu(a+b)k^{(b-1)/2}(2\pi n)}\right]^{1/(a+b)} \tag{5d}$$

where n is the number of cycles of oscillation.

III. Transient Response of Nonlinear Damping

When we use the solution for the nonlinear damping system in Eqs. (1a–1c), it can be shown that the logarithmic decrement δ, which is the ratio of two successive amplitudes, is given by

$$\delta = \frac{1}{a+b}\ell n\left(1 + \frac{T}{t_1 + t_c}\right) \tag{6a}$$

In Eq. (6a), t_1 is the time when the response is first measured to compute the amplitude ratio δ, and T is the period of oscillation. Assuming $t_1 = 0$ and using Eq. (5b) for t_c, we obtain

$$\delta = \frac{1}{a+b}\ell n\left(1 + \frac{2\pi\mu c(a+b)\omega^{b-1}A_0^{a+b}}{m}\right) \tag{6b}$$

For linear viscous damping, it is well known that the logarithmic decrement[10] is given by

$$\delta = \ell n\left(\frac{t_1}{x(t_1+T)}\right) = 2\pi\zeta \tag{6c}$$

Comparing Eq. (6b) with Eq. (6c), one may conclude that while the rate of amplitude decay depends only on the damping ratio ζ, for the linear damping model, the amplitude associated with the nonlinear damping system decreases more rapidly as 1) the initial amplitude A_0 increases; 2) the frequency increases; 3) $a+b$, especially b, increases; and 4) damping ratio ζ increases.

Some numerical results are summarized in Figs. 1–4, which contain the time histories for the nonlinear and linear damping models ($a=b=0$) and for the single-degree-of-freedom system with mass $m=1$, stiffness $k=4$, and damping coefficient $c=0.01$.

For the nonlinear damping cases, $a+b=1$ in both Figs. 1 and 2, but $a+b=2$ in Fig. 3. The initial amplitude A_0 is 25 in Figs. 1 and 3 and 50 in Fig. 2. It can be seen that while the amplitude decrement remains constant for all of the linear damping cases in Figs. 1–3, it decreases more rapidly for the nonlinear damping model. This condition is more evident 1)

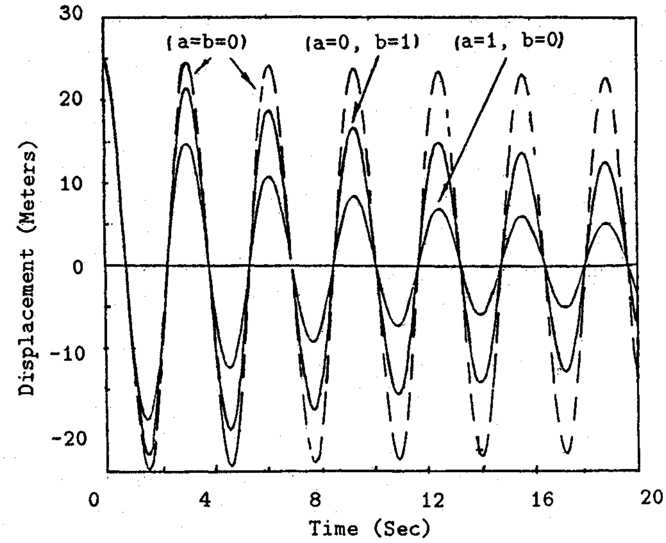

Fig. 1 Time history for linear and nonlinear damping: $a+b=1$, $A_0=25$, mass = 1, stiffness = 4, damping coefficient = 0.01.

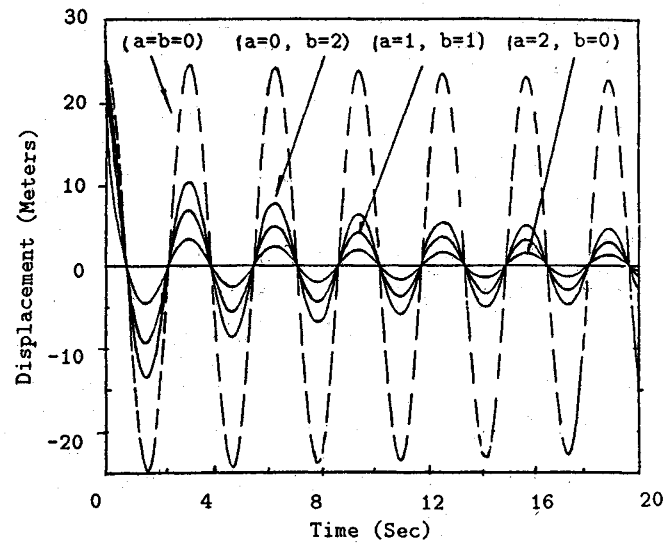

Fig. 3 Time history for linear and nonlinear damping: $a+b=2$, $A_0=25$, mass = 1, stiffness = 4, damping coefficient = 0.01.

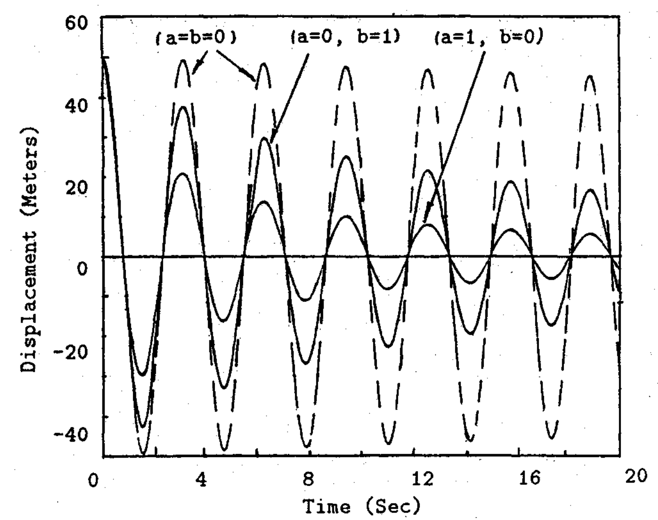

Fig. 2 Time history for linear and nonlinear damping: $a+b=1$, $A_0=50$, mass = 1, stiffness = 4, damping coefficient = 0.01.

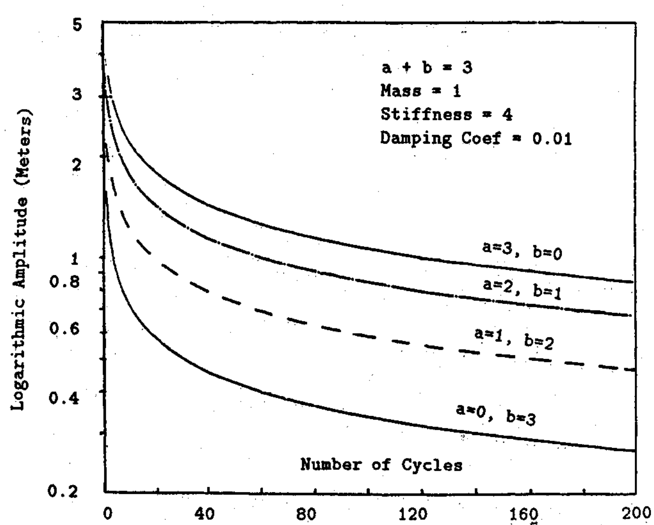

Fig. 4 Amplitude vs number of cycles for linear and nonlinear damping.

at the initial time than at a later time, 2) as the initial amplitude increases, and 3) as $a+b$, especially b, increases. This is in agreement with observations made in practical engineering problems.

Figure 4 compares the logarithmic amplitude decay in terms of the number of cycles of vibration represented by Eq. (5d) for a system with the same parameters as in Figs. 1–3. The sum of $a+b$ remains constant ($a+b=3$) as a and b vary individually. It is found that the rate of amplitude decrease is greater as the value of b increases, even though the sum of a and b remains constant.

IV. Perturbation Solutions of Nonlinear Damping for Distributed Systems

Consider the vibration of a pinned-pinned beam with nonlinear damping

$$\rho\ddot{u} + C_{ab}|u''|^a|\dot{u}''|^b\dot{u}'' + EIu^{iv} = 0 \qquad (7a)$$

$$u(0,t) = u(L,t) = u''(0,t) = u''(L,t) = 0 \qquad (7b)$$

$$u(x,0) = A\sin\frac{m\pi x}{L} \qquad (7c)$$

where ρ is the mass per unit length, L is the length of the beam, and C_{ab} is the nonlinear damping constant.

In Ref. 6, Balakrishnan applied the Krylov-Bogoliubov method to a multidimensional system and obtained expressions similar to those in the single mode case. A common approach to treat an elastic system, such as the pinned-pinned beam discussed here, is to use a modal expansion method to convert a partial differential equation into a series of ordinary differential equations; however, great difficulties were encountered in the modal expansion due to the presence of the absoulte value function in Eq. (7a). For this reason, both an ad hoc perturbation technique and the finite difference method presented in Secs. IV and V were used to study the effects of nonlinear damping for a pinned-pinned beam.

For small values of nonlinear damping coefficient C_{ab}, the system will oscillate at the frequency

$$\omega_m = (m\pi/L)^2\sqrt{(EI)/\rho} \qquad (7d)$$

and period

$$p_m = 2\pi/\omega_m \qquad (7e)$$

Let the perturbation solution after one period be

$$u(x,p_m) = u^0(x,p_m) + \Delta u(x,p_m)$$

where $u^0(x,t)$ is the unperturbed solution of Eqs. (7a–7c) with $C_{ab} = 0$, and $\Delta u(x,t)$ the perturbed solution. In other words

$$u^0(x,p_m) = A \sin\frac{m\pi x}{L} \tag{8a}$$

$$\Delta u(x,p_m) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi x}{L} \tag{8b}$$

The values of the coefficients A_k in Eq. (8b) indicate the degree to which the initial mode is damped out and the other modes are excited. The perturbed solution in Eq. (8b) is given approximately by

$$\Delta\ddot{u} = \frac{C_{ab}}{\rho} |u''|^a |\dot{u}''|^b \dot{u}'' \tag{9a}$$

where $u(x,t)$ on the right-hand side of Eq. (9a) being further approximated by $u^0(x,t)$; that is,

$$u(x,t) = u^0(x,t) = A \sin\frac{m\pi x}{L} \cos\omega_m t \tag{9b}$$

hence

$$u''(x,t) = -A\left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L} \cos\omega_m t \tag{9c}$$

Substitution of Eqs. (9b) and (9c) into Eq. (9a) gives

$$\Delta\ddot{u} = \beta_m \left| \sin\frac{m\pi x}{L}\right|^{a+b} \sin\frac{m\pi x}{L} | \cos\omega_m t|^a \cdot |\sin\omega_m t|^b \sin\omega_m t \tag{9d}$$

where β_m is a constant that depends on many of the system parameters such as the nonlinear damping coefficient C_{ab}, the amplitude A, the mode number m, and so forth; that is,

$$\beta_m \triangleq \beta_m(a,b) = \frac{C_{ab}}{\rho} A^{a+b+1} \omega_m^{b+1} \left(\frac{m\pi}{L}\right)^{2f}$$

where $f \triangleq a + b + 1$.

The coefficient β_m is proportional to $2(a + b + 1)$ powers of the mode number m; hence, $\Delta\ddot{u}$ in Eq. (9d) may become quite large for higher modes. This suggests that the perturbation solution may apply only to low-frequency modes and not to higher frequency modes.

Because we are mainly interested in the maximum perturbation of the amplitude decay due to nonlinear damping after half a period, we integrate Eq. (9d) twice from 0 to t ($t \leqslant p_m/2$) to obtain

$$\Delta u\left(x, \frac{p_m}{2}\right) = \beta_m C_T \left|\sin\frac{m\pi x}{L}\right|^{a+b} \sin\frac{m\pi x}{L} \tag{9e}$$

where

$$C_T \triangleq \int_0^{p_m/2} \mathrm{d}t \int_0^t | \cos\omega_m t|^a | \sin\omega_m t|^b \sin\omega_m \tau \, \mathrm{d}\tau$$

The quantity C_T is a constant that depends on the half period p_m as well as a and b, but does not depend on x. To examine the interaction between different modes in $\Delta u(x,p_m/2)$ in Eq. (9e), we expand $\Delta u(x,p_m/2)$ into a sine Fourier series in terms of the multiples of the initial mode $\sin m\pi x/L$, that is

$$\Delta u\left(x, \frac{p_m}{2}\right) = \sum_{n=1}^{\infty} A_{nm} \sin\frac{nm\pi x}{L} \tag{9f}$$

It can be shown that if the pinned-pinned beam in Eq. (7) is excited only by a single mode $A \sin m\pi x/L$, then for any positive numbers a and b, all of the even multiples of the initial mode will not be excited; only the odd multiples of the initial mode get excited. In other words

$$A_{nm} = 0, \qquad \text{for } n = 2j$$

$$A_{nm} = \frac{2\beta_m C_T}{L} \int_0^L \left|\sin\frac{m\pi x}{L}\right|^{a+b} \sin\frac{m\pi x}{L} \sin\frac{nm\pi x}{L} \, \mathrm{d}x \qquad \text{for } n = 2j - 1$$

Now expand $\Delta u(x,p_m/2)$ into a sine Fourier series to obtain the Fourier coefficients A_{nm}, as represented in Eq. (9f).

It can be shown that for $a = 0$, $b = 1$

$$\begin{aligned} A_{nm} &= 0, && \text{for } n = 2j \\ A_{nm} &= \frac{2C_{ab}\pi^5 m^4 A^2}{[(2j-3)(2j-1)(2j+1)]\rho L^4}, && \text{for } n = 2j - 1 \end{aligned} \tag{10}$$

Similar expressions can be derived for any positive integers a and b. Equation (10) relates the initial mode $A \sin m\pi x/L$ to the distributed nonlinear damping coefficient A_{nm}. Because the perturbation solution contains components of other modes, they too will be excited. The degree of excitation is illustrated by the values of A_{nm}. It is obvious that the even multiples of initial modes are not excited while all of the odd multiples are. It will be shown later that for a beam excited by fundamental modes, Eq. (10) provides an estimate for the amplitude decay after a half period, due to nonlinear damping.

V. Finite Difference Simulation for Nonlinear Damping

In this section, the transversal vibration of a pinned-pinned beam excited by a single mode is simulated via the finite difference method. We will present a finite difference scheme for solving the partial differential equation, derive the stability conditions, and then discuss numerical results. Specifically, we will compute amplitude decay after half a period and compare the results with both linear and nonlinear damping. We treat only the case of $a = 0$, $b = 1$ for nonlinear damping. Finally, we compare the amplitude decrement using the finite difference method with the amplitude decrement obtained using the perturbation method presented in Sec. IV.

Finite Difference Formulations

When modeling the nonlinear differential Eq. (7a) with the finite difference method, we represent beam displacement as $u(x_i,t_j) \triangleq u_{ij}$, where x_i is defined as $x_i \triangleq i\,\Delta x$, and t_j is defined as $t_j \triangleq j\,\Delta t$. Both grids Δx and Δt are defined as $\Delta x \triangleq L/m$, and $\Delta t \triangleq T/n$, where m and n are the numbers of grids used for L and T. L is the beam length and T represents the half period for the initial mode. With the preceding definition, the nonlinear finite difference equation [corresponding to Eq. (7a)] for the displacement at x_i, t_{j+1} is given by

$$\begin{aligned} u_{i,j+1} = {} & \gamma(u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}) \\ & + (2u_{i,j} - u_{i,j-1}) + \beta|(u_{i+1,j} - u_{i+1,j-1}) \\ & - 2(u_{i,j} - u_{i,j-1}) + (u_{i-1,j} - u_{i-1,j-1})| \\ & \cdot [(u_{i+1,j} - u_{i+1,j-1}) - 2(u_{i,j} - u_{i,j-1}) \\ & + (u_{i-1,j} - u_{i-1,j-1})] \end{aligned} \tag{11a}$$

If linear damping is assumed ($a = b = 0$), then Eq. (7a) becomes

$$\rho\ddot{u} + c\dot{u}'' + EIu^{iv} = 0 \tag{11b}$$

and the corresponding finite difference equation is

$$u_{i,j+1} = \gamma(u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}) + (2u_{i,j} - u_{i,j-1}) + \alpha[(u_{i+1,j} - u_{i+1,j-1}) - 2(u_{i,j} - u_{i,j-1}) + (u_{i-1,j} - u_{i-1,j-1})] \tag{11c}$$

Finally, if no damping is present, Eq. (11c) is further simplified as

$$u_{i,j+1} = \gamma(u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}) + (2u_{i,j} - u_{i,j-1}) \tag{11d}$$

where

$$\gamma = -\frac{EI\,\Delta t^2}{\rho\,\Delta x^4}, \qquad \beta = -\frac{C_{ab}}{\rho\,\Delta x^4}, \qquad \alpha = -\frac{c\,\Delta t}{\Delta x^2} \tag{11e}$$

Formulas for central differences have been used to approximate both u^{iv} and $\ddot{u}$. Forward differences for t and central differences for x are used to obtain the mixed derivative $\dot{u}''$ in Eqs. (11a), (11c), and (11d). Two fictitious boundary conditions have been created such that the zero moment conditions are satisfied at both ends for the pinned-pinned beam. They are

$$u_{m+1,j} = -u_{m-1,j}, \qquad u_{-1,j} = -u_{1,j}$$

where $j = 0, 1, 2, \ldots, n$.

Stability Conditions

It should be noted that Eqs. (11a), (11c), and (11d) belong to the explicit forms of finite difference formulation for which there always exists a stability problem for the specific finite difference scheme.[11] In other words, when the finite difference scheme is stable, there exists an upper limit to the extent to which any error arising during the simulation can be amplified. This implies that the numerical solution will not diverge. Clearly, stability alone does not necessarily guarantee that the deviation between the true solution to a certain partial differential equation and its finite difference approximation will be small in any sense. Stability only implies the boundedness of the finite difference solution, at a given time, as Δt approaches zero.

In the case of a beam without damping, the equation of motion is

$$\rho\ddot{u} + EIu^{iv} = 0 \tag{12a}$$

Assuming harmonic motion, that is,

$$u(x,t) = e^{j\lambda x}T(t) \tag{12b}$$

If we use the finite difference method, Eq. (12a) can be replaced by

$$T(t+\Delta t) - 2T(t) + T(t-\Delta t) + \gamma[(e^{2j\,\Delta x} + e^{-2j\,\Delta x}) - 4(e^{j\,\Delta x} + e^{-j\,\Delta x}) + 6]T(t) = 0 \tag{12c}$$

Since

$$[e^{2j\,\Delta x} + e^{-2j\,\Delta x} - 4(e^{j\,\Delta x} + e^{-j\,\Delta x}) + 6] = 16\cos^4(\Delta x/2)$$

Eq. (12c) is simplified to obtain

$$T(t+\Delta t) - [2 - 16\gamma\cos^4(\Delta x/2)]T(t) + T(t-\Delta t) = 0 \tag{12d}$$

Substituting γ in Eq. (11e) into Eq. (12d) and using the stability criteria for the eigenvalues of a difference equation, we find the stability condition for the finite difference Eq. (11d) to be

$$\frac{EI\,\Delta t^2}{\rho\,\Delta x^4}\cos^4\left(\frac{\Delta x}{2}\right) \leqslant \frac{1}{4} \tag{12e}$$

or

$$\frac{EI\,\Delta t^2}{\rho\,\Delta x^4} \leqslant \frac{1}{4} \tag{12f}$$

For a beam vibrating with linear damping, represented by Eq. (11b), the difference equation for the stability condition is

$$T(t+\Delta t) - [2 - 16\gamma\cos^4(\Delta x/2) - 4\alpha\sin^2(\Delta x/2)]T(t) + [1 - 4\alpha\sin^2(\Delta x/2)]T(t-\Delta t) = 0 \tag{13a}$$

This can be further simplified to

$$\frac{4EI\,\Delta t^2}{\rho\,\Delta x^4}\cos^4\left(\frac{\Delta x}{2}\right) + \frac{2\zeta\sqrt{EI\rho}\,\Delta t}{\rho\,\Delta x^2}\sin^2\left(\frac{\Delta x}{2}\right) \leq 1$$

or

$$\frac{EI\,\Delta t^2}{\rho\,\Delta x^4} + \frac{\zeta\sqrt{EI\rho}\,\Delta t}{2\rho\,\Delta x^2} \leq \frac{1}{4} \tag{13b}$$

which is similar to Eq. (12f).

Notice that the stability condition similar to Eqs. (12f) and (13b) is not available for the nonlinear difference equation. This is mainly because of the difficulties encountered due to the presence of the absolute value function in Eq. (11a).

Numerical Examples

The numerical example used for the simulation is a pinned-pinned beam that resembles the SCOLE project mast[1] with $L = 130$ ft, $EI = 4\text{E}07$ lb · ft², $\rho = 0.09556$ slug/ft. The beam is initially at rest and is excited by a single mode $A\sin i\pi x/L$ with $A = 1.3$ ft and $i = 1,2,3,4$. The first frequency is 11.95 rad/s and its corresponding period is 0.5258 s. We proceed to evaluate the displacement of the beam for the first half-period T.

Because the stability condition (12f) implies that Δx cannot be arbitrarily small, we first chose $m = 10$ ($\Delta x = L/m = 13$ ft) when verifying the stability condition. It was found that if $n > 64$, $\Delta t = T/n < 0.0041$, $|\gamma| = 0.2473 < 0.25$, then the numerical scheme was always stable. On the other hand, if $n = 63$, $|\gamma| = 0.2552 > 0.25$, the numerical scheme was found to be unstable, which confirms our stability criteria represented by Eq. (12f).

Before we compare the amplitude decrement using the perturbation and the finite difference methods, it is necessary to verify the finite difference scheme used in the simulation. We compare the finite difference displacement results with those obtained using an analytical solution after a half-period for a beam vibrating without damping [corresponding to Eq. (12a)]. We also compare the finite difference results with those for the logarithmic decrements of the amplitude for a beam vibrating with linear viscous damping [corresponding to Eq. (11b)]. We believe that as long as the finite difference scheme converges reasonably well for both Eqs. (12a) and (11b), the scheme should work for Eq. (11a) with nonlinear damping.

The relative errors were computed for the displacement of a beam vibrating without damping after half a period when 20 intervals in L and 1100 intervals in T were used. The relative errors are defined as the difference between the displacement using the finite difference method and that using an analytical method. The difference is then normalized by the analytical values. The relative errors are quite uniform for each mode over all locations of the beam with their maximum values being 0.0025, 0.037, 0.23, and 0.54% for modes 1, 2, 3, and 4, respectively. This implies that the finite difference

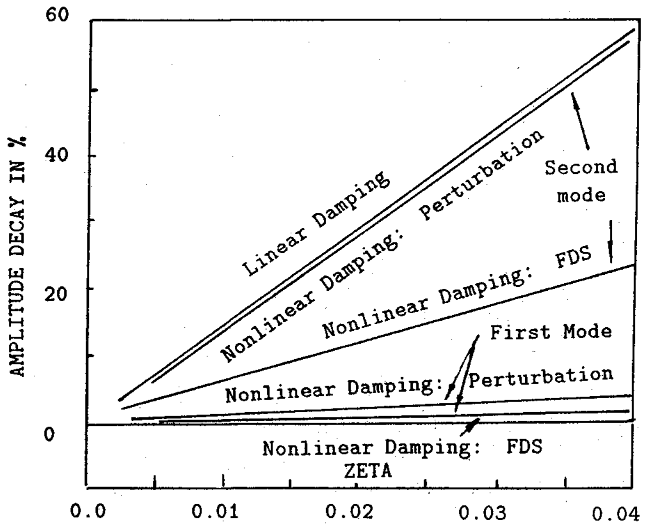

Fig. 5 Amplitude decay vs damping ratio after half-period; FDS: finite difference simulation.

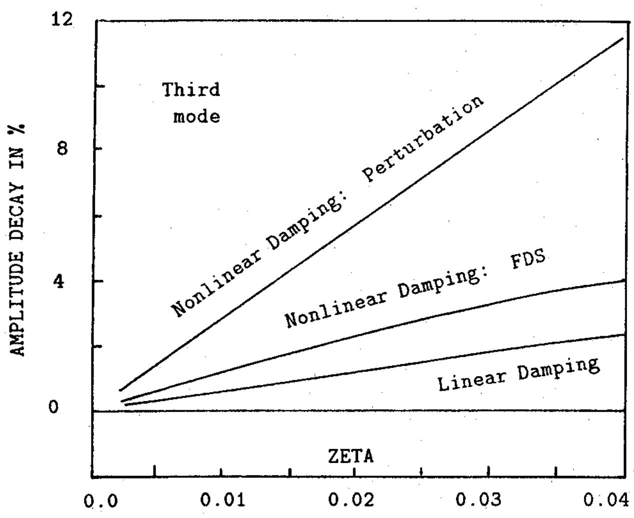

Fig. 6 Amplitude decay vs damping ratio after half-period; FDS: finite difference simulation.

method provides quite accurate results for beam vibrations without damping. Notice that the errors can be greatly reduced if more intervals are used for L and T, subject to the stability condition of Eq. (12f).

For free vibration of a pinned-pinned beam with linear viscous damping, represented by Eq. (11b), it can be shown that by assuming harmonic motion [as in Eq. (12b)], the damping ratio ζ, for each mode depends only on damping constant c, beam properties ρ, and EI, but not on the mode number i, that is,

$$\zeta = c/(2\sqrt{\rho EI}) \tag{14a}$$

According to classical vibration theory,[10] it is well known that the amplitude decay after half a period for single-degree-of-freedom system is

$$\delta \triangleq \frac{\Delta A}{A} = \pi\zeta \tag{14b}$$

which also does not depend on the mode number i. Specifically, if $\zeta = 0.003$, then theoretically δ should equal to 0.0094 after half a period for any single-degree-of-freedom system. Using the same numbers of grid points for L and T as before, the amplitude decay δ, after half a period for modes 1 and 2 at a variety of locations on a beam with linear damping ($\zeta = 0.003$) were computed. For both modes 1 and 2, the amplitude decay δ remains almost identical at all locations of the beam, as if each point on the beam vibrated as a single-degree-of-freedom system. It is interesting to note that the amplitude decay δ is about 0.0094 for mode 1, and 0.0096–0.0097 for mode 2, both of which are close to 0.0094 ($=\pi\zeta$). These results are in good agreement with Eqs. (14a) and (14b) and further confirm that the finite difference algorithm used is quite reliable.

Figures 5 and 6 compare the values of the amplitude decay for modes 1–3 after half a period T for the same beam with 1) linear damping using the finite difference method, 2) nonlinear damping using the finite difference method, and 3) the perturbation method discussed in Sec. IV. The abscissas are a dimensionless quantity damping ratio ζ. Linear damping coefficient c is equal to $2\zeta\sqrt{\rho EI}$, and the nonlinear damping coefficient C_{ab} is equal to $4c$. These values are used for both finite difference and perturbation methods. For the case of linear damping, the amplitude decay takes almost the same value for modes 1–3 (e.g., $\delta = 0.0064$ for modes 1–3 when $\zeta = 0.02$), which is consistent with Eq. (14a).

From Figs. 5 and 6 we observed that 1) the effect of linear damping exceeds that of nonlinear damping for mode 1, 2) the effect of linear damping is about the same as that of nonlinear damping for mode 2, and 3) the effect of nonlinear damping outweighs that of linear damping for mode 3. These results imply that, in the case of nonlinear damping, the amplitude of higher modes will be damped out faster than the lower modes, whereas in the case of linear damping, the modes are all equally damped. This is consistent with the results of Sec. III for a single-degree-of-freedom system, as well as with common engineering judgment.

Finally, when comparing the results in Figs. 5 and 6 for nonlinear damping using either finite difference or perturbation methods, it is observed that, for fundamental modes, the perturbation method provides a very conservative upper bound of amplitude decrement after half a period. This result might be useful in preliminary assessment of the impact of nonlinear damping effect, since in many occasions, we are mainly concerned with the system's fundamental frequencies.

VI. Conclusions

A space system's dynamics and controls analyst is often confronted with the problem of gaining a better understanding of the damping mechanism, which is inherently nonlinear. Fortunately, some of the difficulty in handling nonlinearities is offset by the fact that damping is still small. This makes it possible to obtain approximate solutions using the classical Krylov-Bogoliubov averaging technique to study a class of nonlinear damping models.

In this paper, the damping force is assumed to be proportional to the product of positive integer or fractional powers of the absolute values of displacement and velocity. As is expected for a typical nonlinear system, the amplitude decrement of free vibration with nonlinear models depends not only on damping ratio, but also on the initial amplitude, the time to measure the response, the frequency of the system, and the powers of displacement and velocity. For a pinned-pinned beam, both an ad hoc perturbation method and a finite difference technique are used to study the vibration of a beam with nonlinear damping. The action of nonlinear damping is found to reduce the energy of the system as well as to pass energy to lower modes. As a result, the amplitude of higher modes will be damped out faster than the lower modes. All of these results are very useful to study the response of a flexible structure to the action of a nonlinear boundary feedback control.

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